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On the thermodynamic limit in random resistors networks

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Abstract. We study a random resistors network model on a Euclidean geometry $\boldsymbol{Z}^d$. We formulate the model in terms of a variational principle and show that, under appropriate boundary conditions, the thermodynamic limit of the dissipation per unit volume is finite almost surely and in the mean. Moreover, we show that for a particular thermodynamic limit the result is also independent of the boundary conditions.

1. Introduction

In this paper we study a model of random resistors networks (RRN) on a Euclidean geometry $\boldsymbol{Z}^d$. RRN are examples of disordered statistical mechanical systems which have been widely considered in the literature in the context of percolation theory [1], with different lattice geometries and different probability distributions for the resistors [2–6].

The problem we address is the behaviour of the RRN model in the thermodynamic limit, concentrating in particular on the role of the boundary conditions. We shall restrict ourselves to the case in which the resistors are independent and identically distributed positive random variables. We do not specify any distribution function but assume it to be smooth enough to have a finite expectation value. The physical observable we consider is the dissipation per unit volume, which is related to the total conductance of the network (cf (14)). We show that, under appropriate boundary conditions (which we call *closed boundary conditions* (CBC) and will be specified in section 2), the thermodynamic limit of the dissipation is finite, both in the mean and almost surely. More precisely, the main results of this paper are summarized by the following theorem.

Theorem 1.1. If the conductances of the network have a finite expectation value $\langle C\rangle$, then the limit of the dissipation per unit volume W_{LA}^{CBC}/LA on a rectangular§ RRN of dimensions (L, A),

$$\lim_{L,A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}} \tag{1}$$

exists in the mean and almost surely and is finite, independently of the order of the limits on L and A, where v_0 is a real positive number,

$$v_0^2 \bar{c}^{\mathrm{CBC}} = \lim_{L,A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{LA} \tag{2}$$

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§ We consider the case of $d = 2$, being the extension to the case of an arbitrary d trivial.

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and CBC denotes closed boundary conditions. Moreover, if we let $A \to \infty$ before $L \to \infty$, then

$$\lim_{L\to\infty}\lim_{A\to\infty}\frac{W_{LA}}{LA} = v_0^2\bar{c}^{\mathrm{CBC}} \tag{3}$$

exists in the mean and almost surely and is finite, independently of the boundary conditions.

This paper is organized as follows. In section 2 we define the RRN model based on the classical laws of Ohm and Kirchoff. In section 3 we show how to formulate the model based on a variational principle and in section 4 we use this principle to study the thermodynamic limit of the dissipation per unit volume.

2. The model

We consider a RRN model with an Euclidean geometry $\boldsymbol{Z}^d$ and denote with $R_{nn'}$ the resistors on the links (n, n') of $\boldsymbol{Z}^d$, which are taken to be independent and identically distributed random variables. The model is defined on a finite set $\Lambda \subset \boldsymbol{Z}^d$. We fix a direction, e.g. the direction 1, along which a potential difference is established between the $(d-1)$-dimensional hypersurfaces which are perpendicular to the direction 1. Equally valid RRN can be obtained with current generators instead of potential generators [3].

Boundary conditions (BC) play a fundamental role in all statistical systems [7, 8]. The boundary of the finite region $\Lambda \subset \boldsymbol{Z}^d$ is defined as $\partial\Lambda = \{n \in \Lambda : \exists n',\ n' \notin \Lambda,\ |n - n'| = 1\}$. The BC we use are imposed by fixing the potential on $\partial\Lambda$ such that

$$V_n = n_1 v_0 \qquad n = \{n_1, \dots, n_d\} \in \partial\Lambda \tag{4}$$

where v_0 is a real positive number. If we indicate the set of links contained in Λ with $L_\Lambda = \{(n, n') \subset \Lambda : |n - n'| = 1\}$, we can assign on L_Λ the *conductance field* $C : L_\Lambda \to \boldsymbol{R}$, where $C_{nn'}$ are non-negative, independent and identically distributed random variables, independent of the link orientation, i.e. $C_{nn'} = C_{n'n}$, $(n, n') \in L_\Lambda$. We denote with $(\Omega, \mathcal{F}, \boldsymbol{P})$ the probability space on which the variables $C_{nn'}$ live, and with $\boldsymbol{E}(\cdot)$ the expectation value respect to the measure $\boldsymbol{P}$. We use the notation $\boldsymbol{E}(C^k_{nn'}) = \langle C^k\rangle$, $k = 1, 2, \dots$, $(n, n') \in L_\Lambda$, for the expectation value of the kth power of the variables $C_{nn'}$.

The *potential field* $V : \Lambda \to \boldsymbol{R}$ is related to the conductance field C by Ohm's law

$$(V_n - V_{n'})C_{nn'} = I_{nn'} \qquad (n, n') \in L_\Lambda \tag{5}$$

where $I_{nn'}$ is the current passing though the link (n, n'). Denoting with $\bar{\Lambda} \subset \Lambda$ the set of points of Λ on which the potential field V is fixed to a value $\bar{V}$, Kirchoff's First Law is given by

$$\sum_{\{n'\in\Lambda:|n'-n|=1\}} I_{nn'} = \sum_{\{n'\in\Lambda:|n'-n|=1\}} (V_n - V_{n'})C_{nn'} = 0 \qquad n \in \Lambda\backslash\bar{\Lambda}. \tag{6}$$

Thus, by Ohm's and Kirchoff's laws, on the sites $n \in \Lambda\backslash\bar{\Lambda}$ there is a well-defined potential

$$V_n = \frac{\sum_{\{n'\in\Lambda:|n'-n|=1\}} V_{n'}C_{nn'}}{\sum_{\{n'\in\Lambda:|n'-n|=1\}} C_{nn'}} \qquad n \in \Lambda\backslash\bar{\Lambda}. \tag{7}$$

Note that Kirchoff's and Ohm's laws are relations valid for each $\omega \in \Omega$. Proposition 2.1 will be useful in the following.

Proposition 2.1 (maximum principle). Let $\bar{V}_n$, $n \in \bar{\Lambda}$ the values of the fixed potential and $V_{\max}$, $V_{\min}$ its maximum and minimum values, $V_{\min} \leqslant \bar{V}_n \leqslant V_{\max}$, $n \in \bar{\Lambda}$. Then, for each $\omega \in \Omega$, $V_{\min} \leqslant V_n \leqslant V_{\max}$, $n \in \Lambda\backslash\bar{\Lambda}$.

Proof. Let us show, for example, the inequality $V_n \leqslant V_{\max}$. By equation (7), the potentials V_n, $n \in \Lambda\backslash\bar{\Lambda}$, are given by a weighted average of $V_{n'}$ on the nearest-neighbour sites $n' \in \Lambda$, i.e. for every $\omega \in \Omega$

$$V_n \leqslant \max_{\{n':|n'-n|=1\}} V_{n'} \qquad n \in \Lambda\backslash\bar{\Lambda},\ n' \in \Lambda. \tag{8}$$

If there exists a point $\bar{n} \in \Lambda\backslash\bar{\Lambda}$ such that $V_{\bar{n}} > V_{\max}$, then we would have $V_{\bar{n}} > \bar{V}_{n'}$, for every $n' \in \bar{\Lambda}$. On the other hand, repeatedly applying (8) we can find a $n' \in \bar{\Lambda}$ such that $V_{\bar{n}} \leqslant \bar{V}_{n'}$, hence the contradiction.

3. Variational principle

In this section we show that Kirchoff's law, (cf (6)), can be obtained by a variational principle. In particular, the potential field V determined by Kirchoff's law minimize the dissipation per unit volume by the Joule effect by the network in the region Λ

$$w_\Lambda(C,\bar{V}) = \frac{W_\Lambda(C,\bar{V})}{|\Lambda|} = \frac{1}{|\Lambda|}\sum_{(n,n')\in L_\Lambda}(V_n - V_{n'})^2 C_{nn'} \tag{9}$$

where $\bar{V}$ denotes a field of fixed values for the potential on $\bar{\Lambda}$ and the sum is taken over all the links (n,n') in the region Λ†. We introduce the set of the potential fields which coincide with $\bar{V}$ on $\bar{\Lambda}$, $\Phi(\bar{\Lambda},\bar{V}) = \{\phi : \phi_n = \bar{V}_n,\ n \in \bar{\Lambda}\}$, and the *dissipation functional* $\varphi_\Lambda : \Phi(\bar{\Lambda},\bar{V}) \to \boldsymbol{R}$

$$\varphi_\Lambda(C,\phi) = \sum_{(n,n')\in L_\Lambda}(\phi_n - \phi_{n'})^2 C_{nn'}. \tag{10}$$

Definition 3.1. Given a field $\phi \in \Phi(\bar{\Lambda},\bar{V})$, the function $U_n^\epsilon : \Lambda \times [-1,1] \to \boldsymbol{R}$ is called the variation of the field ϕ in $\Phi(\bar{\Lambda},\bar{V})$ if

(i) $U_n^{\epsilon=0} = \phi_n$, for every $n \in \Lambda$,

(ii) $U_n^\epsilon = \bar{V}_n$, for every $n \in \bar{\Lambda}$ and $\epsilon \in [-1,1]$,

(iii) $U^\epsilon \in C^\infty(\Lambda \times [-1,1])$.

We denote with $\mathcal{V}(\phi)$ the set of variations of ϕ in $\Phi(\bar{\Lambda},\bar{V})$.

Definition 3.2. $V \in \Phi(\bar{\Lambda},\bar{V})$ is called a stationary point for $\varphi_\Lambda(C,\phi)$ on $\Phi(\bar{\Lambda},\bar{V})$ if the function $\varphi_\Lambda(C,U^\epsilon) : [-1,1] \to \boldsymbol{R}$ has a stationary point in $\epsilon = 0$ for every $U^\epsilon \in \mathcal{V}(V)$.

We can now prove the main result of this section:

Theorem 3.1 (least dissipation principle). The potential field $V \in \Phi(\bar{\Lambda},\bar{V})$ determined by Kirchoff's law is a minimum point for the functional $\varphi_\Lambda(C,\phi)$, i.e.

$$W_\Lambda(C,\bar{V}) = \min_{\{\phi:\phi_n=\bar{V}_n,\ n\in\bar{\Lambda}\}} \varphi_\Lambda(C,\phi). \tag{11}$$

Proof. We want to show that the potential field $V \in \Phi(\bar{\Lambda},\bar{V})$ determined by Kirchoff's law is both a stationary point and a minimum point for the dissipation functional $\varphi_\Lambda(C,\phi)$.

The stationarity is proven by noting that for every $U^\epsilon \in \mathcal{V}(V)$, $V \in \Phi(\bar{\Lambda},\bar{V})$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\varphi_\Lambda(C,U^\epsilon)\right|_{\epsilon=0} = 2\sum_{n\in\Lambda\backslash\bar{\Lambda}}\Bigg(\sum_{\{n'\in\Lambda:|n'-n|=1\}}(V_n - V_{n'})C_{nn'}\Bigg)Z_n \tag{12}$$

† We stress that notation $(n,n') \in L_\Lambda$ should be intended as a sum over the number of links and not simply as a sum over nearest-neighbour points n and n'. The latter sum differs from the former by a factor of 2.

having used the fact that, by definition 3.1, $Z_n = \partial U_n^\epsilon/\partial\epsilon|_{\epsilon=0} = 0$ for every $n \in \bar{\Lambda}$. Then, by Kirchoff's law (6) and the arbitrariness of the Z_n, $n \in \Lambda\backslash\bar{\Lambda}$, consequence of the arbitrariness of the $U^\epsilon \in \mathcal{V}(V)$, definition 3.2 implies that $V \in \Phi(\bar{\Lambda}, \bar{V})$ is a stationary point for $\varphi_\Lambda(C, \phi)$.

To prove that the potential field $V \in \Phi(\bar{\Lambda}, \bar{V})$ is a minimum point, we need to show that $\varphi_\Lambda(C, U^\epsilon) \geqslant \varphi_\Lambda(C, V)$, for every $U^\epsilon \in \mathcal{V}(V)$ and every $\epsilon \in [-1, 1]$. Putting $U_n^\epsilon = V_n + Z_n^\epsilon$ we obtain

$$\begin{aligned}\varphi_\Lambda(C, U^\epsilon) - \varphi_\Lambda(C, V) &\geqslant 2 \sum_{(n,n')\in L_\Lambda} (V_n - V_{n'})(Z_n^\epsilon - Z_{n'}^\epsilon)C_{nn'} \\ &= 2\sum_{n\in\Lambda}\Bigg(\sum_{\{n'\in\Lambda:|n'-n|=1\}} (V_n - V_{n'})C_{nn'}\Bigg)Z_n^\epsilon.\end{aligned}$$

Then, using Kirchoff's law (6), we get $\varphi_\Lambda(C, U^\epsilon) - \varphi_\Lambda(C, V) \geqslant 0$, for each Z_n^ϵ, i.e. for every $\epsilon \in [-1, 1]$.

4. Thermodynamic limit

In this section, we study the thermodynamic limit of the dissipation density $W_\Lambda/|\Lambda|$, as the volume $|\Lambda|$ of the region $\Lambda \subset \boldsymbol{Z}^d$ goes to infinity. We parametrize the region $\Lambda \subset \boldsymbol{Z}^d$ with a rectangle of dimensions $a_1, \ldots, a_d$ and then take the limit $a_i \to \infty$, $i = 1, \ldots, d$. For notational simplicity we shall consider the case with $d = 2$, as the results are trivially extendable to arbitrary d. We parametrize the rectangle by a pair of integers (L, A), where L is the number of links in the longitudinal direction and A is the number of sites in the transverse direction.

We denote the dissipation density in two dimensions with†

$$\frac{W_{LA}}{LA} = \frac{1}{LA}\sum_{(n,n')\in L_\Lambda} (V_n - V_{n'})^2 C_{nn'}. \tag{13}$$

Let us establish the potential difference along the longitudinal direction. We shall consider the case in which no boundary conditions are imposed along the longitudinal direction (*open BC*), and the case in which the boundary conditions are imposed following equation (4) (*closed BC*).

Let us note that, being the potential difference proportional to the longitudinal dimension, the dissipation density can be written in terms of the total conductance C_{LA} of the network,

$$\frac{W_{LA}}{LA} = \frac{V_L^2 C_{LA}}{LA} = v_0^2 \frac{L}{A} C_{LA}. \tag{14}$$

Indeed, the authors of [3] study the thermodynamic limit of the r.h.s. of (14).

4.1. Preliminary lemmas

In this section we apply the variational principle of section 3 to derive a few properties for the dissipation W_{LA}. In particular, we show that W_{LA} is a subadditive and superadditive sequence of random values, with respect to L and A, depending on the BC. We refer the reader to [3, 9, 10], for the definition of subadditivity and superadditivity sequences, while the main theorems we use in this paper are reported in the appendix.

† It must be stressed that, even if the notation might be slightly misleading, the dissipation is a function of the region $\Lambda(L, A)$ of dimensions (L, A) and not only of the dimensions (L, A). It would be more appropriate to write $W_{\Lambda(L,A)}$, but when not strictly necessary for the comprehension, we shall use the abbreviated form.

In the following OBC and CBC denote the open and closed BC respectively, while # both BC.

Lemma 4.1. For every $\omega \in \Omega$, $W_{LA}^{\mathrm{OBC}}(\omega) \leqslant W_{LA}^{\mathrm{CBC}}(\omega)$.

Proof. By the least dissipation principle, theorem 3.1, the total dissipation $W(C, \bar{V})$ is obtained minimizing, for every $\omega \in \Omega$, the functional $\varphi(C, \phi)$ with respect to all test fields ϕ, with the constraint that $\phi_n = \bar{V}_n$, $\forall n \in \bar{\Lambda}$. Since, by imposing more constraints on ϕ, we restrict the space on which they can vary, we have that for every $\omega \in \Omega$ the minimum of $\varphi(C, \phi)$ on the reduced space will be greater or equal to the one on the space with less constraints. Hence, the thesis follows by considering the closed BC as a greater number of constraints with respect to the open BC.

The following properties of subadditivity and superadditivity on W_{LA} hold:

Lemma 4.2. If $\langle C \rangle < \infty$,

(i) $W_{LA}^{\#}$ is subadditive in L for every fixed $A \in \mathbb{N}$,

(ii) W_{LA}^{OBC} is superadditive in A for every fixed $L \in \mathbb{N}$,

(iii) W_{LA}^{CBC} is subadditive in A for every fixed $L \in \mathbb{N}$,

with respect to a translation on the probability space.

Proof. As in the proof of lemma 4.1, the dissipation on a region $\Lambda(L, A)$ is less or equal to the dissipation on the same region on which we impose a greater number of constraints on the test potentials.

To prove point (i), we note that if $\Lambda(L, A) = \Lambda_1(L_1, A) \bigcup \Lambda_2(L_2, A)$ with $L = L_1 + L_2$, then

$$W_{\Lambda(L,A)}^{\#} \leqslant W_{\Lambda_1(L_1,A)}^{\#} + W_{\Lambda_2(L_2,A)}^{\#}$$

since to get the r.h.s. we need to impose the constraint that the potential of the sites with longitudinal coordinate L_1 be $v_0 L_1$. Since the random variables are independent and identically distributed we can introduce the translation operator in the longitudinal direction τ_l on $W_{LA}^{\#} \equiv W_{\Lambda(L,A)}^{\#}$

$$W_{L_1+L_2,A}^{\#} \leqslant W_{L_1 A}^{\#} + W_{L_2 A}^{\#} \circ \tau_l^{L_1}$$

from which the subadditivity of $W_{LA}^{\#}$ with respect to L for fixed $A \in \mathbb{N}$. It is clear that this relation is valid independently of the contraints, i.e. of the BC.

Point (iii) is proven in a similar way, imposing as a constraint the fact that the potential on the sites of the adjacent boundaries of the regions $\Lambda_1'(L, A_1)$ and $\Lambda_2'(L, A_2)$, such that $\Lambda(L, A) = \Lambda_1'(L, A_1) \bigcup \Lambda_2'(L, A_2)$ with $A = A_1 + A_2$, are proportional to the longitudinal coordinate.

Finally, to prove point (ii), we observe that the region $\Lambda(L, A)$ with open BC can be obtained connecting with resistors the nearest-neighbour sites of the adjacent boundaries of two regions $\Lambda_1'(L, A_1)$ and $\Lambda_2'(L, A_2)$, also with open BC. To do this, we must take $\Lambda(L, A) = \Lambda_1'(L, A_1) \bigcup \Lambda_2'(L, A_2)$ with $A = A_1 + A_2$. On the other hand, connecting two sites with a resistor implies passing from a conductance $C = 0$ to a conductance $C \geqslant 0$. In this passage the dissipation cannot diminish, i.e.

$$W_{\Lambda_1'(L,A)}^{\mathrm{OBC}} \geqslant W_{\Lambda_1'(L,A_1)}^{\mathrm{OBC}} + W_{\Lambda_2'(L,A_2)}^{\mathrm{OBC}}.$$

Introducing the translation operator in the transverse direction τ_t, we can write

$$W_{L,A_1+A_2}^{\mathrm{OBC}} \geqslant W_{LA_1}^{\mathrm{OBC}} + W_{LA_2}^{\mathrm{OBC}} \circ \tau_t^{A_1}.$$

from which the superadditivity W_{LA}^{OBC} with respect to A for fixed $L \in \mathbb{N}$.

To complete the proof, we only need to observe that if the expectation value $\langle C\rangle$ is finite, the expectation values $\boldsymbol{E}(W_{LA})/LA$ are also finite for every $L, A \in \mathbb{N}$.

From lemmas 4.2 and A.1, we have the following

Lemma 4.3. If $\langle C\rangle < \infty$

(i) $\lim_{L\to\infty} \frac{1}{L}\boldsymbol{E}(W^{\#}_{LA}) = \inf_{L} \frac{1}{L}\boldsymbol{E}(W^{\#}_{LA})$,

(ii) $\lim_{A\to\infty} \frac{1}{A}\boldsymbol{E}(W^{\mathrm{OBC}}_{LA}) = \sup_{A} \frac{1}{A}\boldsymbol{E}(W^{\mathrm{OBC}}_{LA})$,

(iii) $\lim_{A\to\infty} \frac{1}{A}\boldsymbol{E}(W^{\mathrm{CBC}}_{LA}) = \inf_{A} \frac{1}{A}\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})$.

4.2. Convergence in the mean

In this section we prove the convergence in the mean (mean square convergence) [11] of $(1/LA)W_{LA}$ as $L \to \infty$ and $A \to \infty$. For closed BC we demonstrate the convergence independently of the order of the limits on L and A. For open BC, the result depends on the order of the limits. We shall divide the proof in a few preliminary propositions.

Proposition 4.1. If $\langle C\rangle < \infty$ the limits

(I) $\lim_{A\to\infty} \frac{\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})}{A} = \inf_{A} \frac{\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})}{A} \equiv v_0^2 L^2 \bar{g}^{\mathrm{CBC}}_L$

(ii) $\lim_{A\to\infty} \frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{LA})}{A} = \sup_{A} \frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{LA})}{A} \equiv v_0^2 L^2 \bar{g}^{\mathrm{OBC}}_L$

(iii) $\lim_{L\to\infty} \frac{\boldsymbol{E}(W^{\mathrm{CBC,OBC}}_{LA})}{L} = \inf_{L} \frac{\boldsymbol{E}(W^{\mathrm{CBC,OBC}}_{LA})}{L} \equiv v_0^2 \bar{c}^{\mathrm{CBC,OBC}}_A$

exist and are finite, with $\bar{g}^{\mathrm{OBC}}_L \leqslant \bar{g}^{\mathrm{CBC}}_L$, for every $L, A \in \mathbb{N}$.

Proof. We consider first the case of closed BC. Being the sequence $\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})$ bounded from below by zero, by lemma 4.3 the limit

$$\lim_{A\to\infty} \frac{\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})}{A} = \inf_{A} \frac{\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})}{A} \equiv v_0^2 L^2 \bar{g}^{\mathrm{CBC}}_L$$

exists finite for every $L \in \mathbb{N}$. For open BC, by lemma 4.1,

$$\frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{LA})}{A} \leqslant \frac{\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})}{A}$$

for every $A \in \mathbb{N}$, hence passing to the limit $A \to \infty$, by lemma 4.3 the limit

$$\lim_{A\to\infty} \frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{LA})}{A} = \sup_{A} \frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{LA})}{A} \equiv v_0^2 L^2 \bar{g}^{\mathrm{OBC}}_L$$

exists finite, with $\bar{g}^{\mathrm{OBC}}_L \leqslant \bar{g}^{\mathrm{CBC}}_L$, for every $L \in \mathbb{N}$. Point (iii) is proven along similar lines using the subadditivity in L of $\boldsymbol{E}(W_{LA})$, valid independently of the BC.

Proposition 4.2. If $\langle C\rangle < \infty$ the limits

(I) $\lim_{A\to\infty} \frac{\bar{c}^{\mathrm{CBC}}_A}{A} = \inf_{A} \frac{\bar{c}^{\mathrm{CBC}}_A}{A} \equiv \bar{c}^{\mathrm{CBC}}$,

(ii) $\lim_{A\to\infty} \frac{\bar{c}^{\mathrm{OBC}}_A}{A} = \sup_{A} \frac{\bar{c}^{\mathrm{OBC}}_A}{A} \equiv \bar{c}^{\mathrm{OBC}}$,

(iii) $\lim_{L\to\infty} L\bar{g}^{\mathrm{CBC,OBC}}_L = \inf_{L} L\bar{g}^{\mathrm{CBC,OBC}}_L \equiv \bar{g}^{\mathrm{CBC,OBC}}$,

exist and are finite, with $\bar{c}^{\mathrm{OBC}} \leqslant \bar{c}^{\mathrm{CBC}}$.

Proof. By lemma 4.2 and proposition 4.1, it is easily proven that $\bar{c}_A^{\mathrm{CBC}}$ and $\bar{c}_A^{\mathrm{OBC}}$ are subadditive and superadditive sequences in A, respectively. Then, by lemma A.1, being $\bar{c}_A^{\mathrm{CBC}}$ bounded from below by zero, the limit

$$\lim_{A\to\infty} \frac{\bar{c}_A^{\mathrm{CBC}}}{A} = \inf_A \frac{\bar{c}_A^{\mathrm{CBC}}}{A} \equiv \bar{c}^{\mathrm{CBC}}$$

exists finite. Moreover, by lemma 4.1 and proposition 4.1, $\bar{c}_A^{\mathrm{OBC}} \leqslant \bar{c}_A^{\mathrm{CBC}}$ for every $A \in \mathbb{N}$. Then, passing to the limit for $A \to \infty$, the limit

$$\lim_{A\to\infty} \frac{\bar{c}_A^{\mathrm{OBC}}}{A} = \sup_A \frac{\bar{c}_A^{\mathrm{OBC}}}{A} \equiv \bar{c}^{\mathrm{OBC}}$$

exists finite, with $\bar{c}^{\mathrm{OBC}} \leqslant \bar{c}^{\mathrm{CBC}}$. Point (iii) is similarly proven.

Proposition 4.3. If $\langle C \rangle < \infty$ then

$$\lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{OBC}})}{A} = \lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{A}. \tag{15}$$

Proof. By lemma 4.1, it suffices to show that

$$\lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{A} \leqslant \lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{OBC}})}{A}. \tag{16}$$

To prove this inequality, we consider a network of dimensions (L, A) with closed BC, and a network of dimensions $(L, A-2)$ with open BC, as shown in figure 1.

By the least dissipation principle (cf theorem 3.1), $W_{LA}^{\mathrm{CBC}} \equiv W_{\Lambda(L,A)}^{\mathrm{CBC}}(C, \bar{V})$ is obtained by minimizing the functional $\varphi(C, \phi)$ with respect to all the fields ϕ that assume the values $v_0, 2v_0, \ldots, Lv_0$ on the boundaries parallel to the longitudinal direction. Denoting by V the extremal field, we have

$$W_{LA}^{\mathrm{CBC}} = \sum_{(n,n')\in L_{\Lambda(L,A)}} (V_n - V_{n'})^2 C_{nn'}.$$

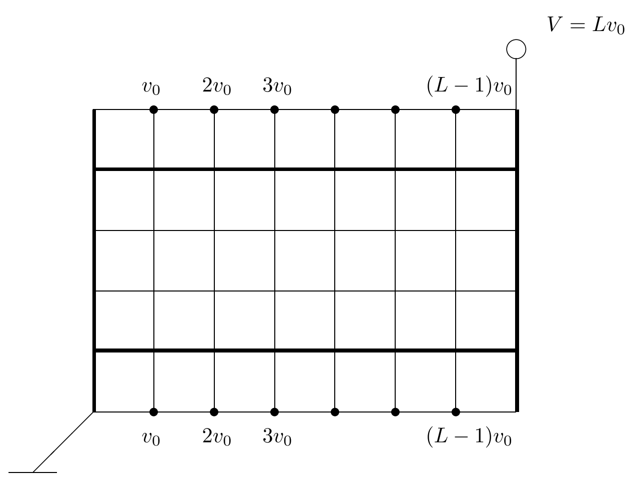

Figure 1. Comparison of the network of dimensions (L, A) with closed BC with the network of dimensions $(L, A-2)$ with open BC.

The extremal field V' for $W^{\mathrm{OBC}}_{L,A-2} \equiv W^{\mathrm{OBC}}_{\Lambda'(L,A-2)}(C, \bar{V}')$ is, in general, not equal to V, i.e. it does not minimize W^{CBC}_{LA}. Thus we have

$$\begin{aligned} W^{\mathrm{CBC}}_{LA} = & \sum_{(n,n')\in L_{\Lambda'(L,A-2)}} (V_n - V_{n'})^2 C_{nn'} + \sum_{(n,n')\in L_{\Lambda(L,A)} \setminus L_{\Lambda'(L,A-2)}} (V_n - V_{n'})^2 C_{nn'} \\ & \leqslant \sum_{(n,n')\in L_{\Lambda'(L,A-2)}} (V'_n - V'_{n'})^2 C_{nn'} \\ & \quad + \sum_{\{n\in\Lambda'(L,A-2), n'\notin\Lambda'(L,A-2): |n-n'|=1\}} (V'_n - n'_1 v_0)^2 C_{nn'} \\ & \quad + \sum_{(n,n')\in L_{\Lambda(L,A)\setminus\Lambda'(L,A-2)}} (n_1 v_0 - n'_1 v_0)^2 C_{nn'}. \end{aligned}$$

from which

$$\frac{\boldsymbol{E}(W^{\mathrm{CBC}}_{LA})}{A} \leqslant \frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{L,A-2})}{A} + \frac{\boldsymbol{E}(W^{\mathrm{tr}}_{LA})}{A} + \frac{\boldsymbol{E}(W^{\mathrm{bound}}_{L})}{A} \tag{17}$$

where

$$W^{\mathrm{bound}}_{L} = v_0^2 \sum_{k=1}^{2L} C^{\mathrm{bound}}_k \tag{18}$$

is the dissipation due to the conductances C^{bound}_k, $k = 1, \ldots, 2L$ which lie on the longitudinal boundaries of the network of dimensions (L, A) with closed BC and

$$W^{\mathrm{tr}}_{LA} = W^{\mathrm{sup}}_{LA} + W^{\mathrm{inf}}_{LA} \tag{19}$$

is the dissipation due to the transverse conductances of $L_{\Lambda(L,A)} \setminus L_{\Lambda'(L,A-2)}$ that do not lie on the equipotential sides, which do not dissipate. The superscripts sup and inf indicate the contributions of the superior and inferior part of $\Lambda(L, A)\setminus\Lambda'(L, A-2)$. We indicate with C^{sup}_k and C^{inf}_k, $k = 1, \ldots, L-1$ these conductances, and with V^{sup}_k and V^{inf}_k, $k = 1, \ldots, L-1$, the potentials on the sites on the two longitudinal boundaries of the network of dimensions $(L, A-2)$ with open BC, excluding the sites that lie on the equipotential sides. Thus, we have

$$W^{\mathrm{sup,inf}}_{L} = v_0^2 \sum_{k=1}^{L-1} (V^{\mathrm{sup,inf}}_k - k v_0)^2 C^{\mathrm{sup,inf}}_k. \tag{20}$$

Note that the transverse term W^{tr}_{LA} in general can depend both on L and A, while the boundary term W^{bound}_{L} does not depend on A, so it does not contribute to the limit for $A \to \infty$ of (17). Thus, we only need to show that

$$\lim_{A\to\infty} \frac{\boldsymbol{E}(W^{\mathrm{tr}}_{LA})}{A} = 0 \tag{21}$$

from which, since

$$\frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{L,A-2})}{A} \leqslant \frac{\boldsymbol{E}(W^{\mathrm{OBC}}_{L,A-2})}{A-2} \tag{22}$$

i.e. equation (16).

To estimate the term $\boldsymbol{E}(W^{\mathrm{tr}}_{LA})$, note that by the maximum principle (cf proposition 2.1), we have $0 \leqslant V^{\mathrm{sup,inf}}_k \leqslant L v_0$, for every $k < L$, from which

$$|V^{\mathrm{sup,inf}}_k - k v_0| \leqslant (L-1) v_0 \tag{23}$$

for every $k < L$. Then, $W_{LA}^{\sup,\inf} \leqslant (L-1)^2 v_0^2 \sum_{k=1}^{L-1} C_k^{\sup,\inf}$ and, being the variables identically distributed,

$$\boldsymbol{E}(W_{LA}^{\mathrm{tr}}) \leqslant 2(L-1)^3 v_0^2 \langle C \rangle. \tag{24}$$

Since this estimate is uniform in A, we obtain (21) and thus the thesis.

We can sum up the preceding propositions in the following.

Theorem 4.1. If $\langle C \rangle < \infty$, the limit

$$\lim_{L,A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}} \tag{25}$$

exist and is finite, independently of the order with which we take the limits, while

$$\lim_{A\to\infty} \lim_{L\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{OBC}})}{LA} = v_0^2 \bar{c}^{\mathrm{OBC}}$$
$$\lim_{L\to\infty} \lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{OBC}})}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}}$$

with $\bar{c}^{\mathrm{OBC}} \leqslant \bar{c}^{\mathrm{CBC}}$.

Proof. By propositions 4.1 and 4.2,

$$\lim_{L\to\infty} \lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC,OBC}})}{LA} = v_0^2 \bar{g}^{\mathrm{CBC,OBC}}$$
$$\lim_{A\to\infty} \lim_{L\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC,OBC}})}{LA} = v_0^2 \bar{c}^{\mathrm{CBC,OBC}}$$

so we only need to show that $\bar{g}^{\mathrm{OBC}} = \bar{g}^{\mathrm{CBC}} = \bar{c}^{\mathrm{CBC}}$. The first equality follows from proposition 4.3, since

$$\bar{g}_L^{\mathrm{CBC}} = \lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{A} = \lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{OBC}})}{A} = \bar{g}_L^{\mathrm{OBC}}$$

for every $L \in \mathbb{N}$. The second equality from the fact that

$$\bar{g}^{\mathrm{CBC}} = \inf_L L \bar{g}_L^{\mathrm{CBC}} = \inf_{L,A} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{LA} = \inf_A \frac{\bar{c}_A^{\mathrm{CBC}}}{A} = \bar{c}^{\mathrm{CBC}}.$$

4.3. Almost sure convergence

In this section we prove the main results of this paper. We first show that, for closed BC, the dissipation density W_{LA}/LA converges almost surely, as $L, A \to \infty$, independently of the order of the limits. This is done by exploiting general theorems on subadditive sequences [9, 3], reported in the appendix. We also show that the almost sure convergence holds for both open and closed BC if we let $A \to \infty$ before $L \to \infty$, i.e. we prove the independence of the boundary conditions for a given order of the limits. The novelty of this second result is that the different behaviour of the dissipation with open and closed BC is exploited to prove the almost sure convergence, using classical theorems of probability theory, such as Kolmogorov's strong law of large numbers [11], instead of the general theorems of the appendix.

4.3.1. Independence of the order of the limits for closed BC.

Theorem 4.2. For closed BC, the limit

$$\lim_{L,A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}} \tag{26}$$

exists almost surely and is finite, independently of the order of the limits, where

$$v_0^2 \bar{c}^{\mathrm{CBC}} = \lim_{L,A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{LA}. \tag{27}$$

Proof. We must show that almost surely the limits

(i) $\displaystyle\lim_{A\to\infty}\lim_{L\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}}$,

(ii) $\displaystyle\lim_{L\to\infty}\lim_{A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}}$,

exist and are finite. By lemma 4.2, W_{LA}^{CBC} is subadditive in L for every fixed A, and in A for every fixed L, with a translation as a measure-preserving transformation on the probability space. Hence, by the theorems in the appendix and proposition 4.1, we get that the limits

$$\lim_{L\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = \inf_L \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{LA} = v_0^2 \bar{c}_A^{\mathrm{CBC}}$$

$$\lim_{A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = \inf_A \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{LA} = v_0^2 L \bar{g}_L^{\mathrm{CBC}}$$

exist almost surely and are finite. Then, by proposition 4.2, the limits

$$\lim_{A\to\infty}\lim_{L\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}}$$

$$\lim_{L\to\infty}\lim_{A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}}{LA} = v_0^2 \bar{g}^{\mathrm{CBC}}$$

also exist almost surely and are finite. The thesis follows from theorem 4.1.

4.3.2. Independence of the boundary conditions for a given order of the limits.

Theorem 4.3. Independently of the BC, the limit

$$\lim_{L\to\infty}\lim_{A\to\infty} \frac{W_{LA}}{LA} = v_0^2 \bar{c}^{\mathrm{CBC}} \tag{28}$$

exists almost surely and is finite, where

$$v_0^2 \bar{c}^{\mathrm{CBC}} = \lim_{L\to\infty}\lim_{A\to\infty} \frac{\boldsymbol{E}(W_{LA}^{\mathrm{CBC}})}{LA}. \tag{29}$$

Proof. Let us consider an increasing sequence of integers $\{A_p\}$ such that $A_p \to \infty$, $p \to \infty$. We fix our attention on a generic element of the sequence A_p such that $A = N_p A_p + r_p$, $r_p < A_p$. Let us divide the network of transverse extension A in N_p networks of extension A_p plus a network of extension r_p. Being W_{LA}^{OBC} a superadditive sequence in A for every fixed $L \in \mathbb{N}$, we have

$$W_{LA}^{\mathrm{OBC}} \geqslant \sum_{k=1}^{N_p} W_{LA_p}^{\mathrm{OBC}} \circ \tau^{(k-1)A_p} + W_{Lr_p}^{\mathrm{OBC}} \circ \tau^{NA_p} \tag{30}$$

where $W_{LA_p}^{\mathrm{OBC}} \circ \tau^{(k-1)A_p}$, $k = 1, \ldots, N_p$, are independent and identically distributed random variables. Dividing both sides by $A - r_p = N_p A_p$, and letting $A \to \infty$, or equivalently $N_p \to \infty$,

$$\liminf_{A\to\infty} \frac{W_{LA}^{\mathrm{OBC}}}{A} \geqslant \lim_{N_p\to\infty} \frac{1}{N_p} \sum_{k=1}^{N_p} \frac{W_{LA_p}^{\mathrm{OBC}} \circ \tau^{(k-1)A_p}}{A_p}.$$

Since the expectation value $(1/A_p)\boldsymbol{E}(W_{LA_p}^{\mathrm{OBC}})$ is finite we can apply Kolmogorov's strong law of large numbers [11] to state that $\boldsymbol{P}(\mathcal{N}_p^{\mathrm{OBC}}) = 0$, where

$$\mathcal{N}_p^{\mathrm{OBC}} = \left\{ \liminf_{A\to\infty} \frac{W_{LA}^{\mathrm{OBC}}(\omega)}{A} < \frac{\boldsymbol{E}(W_{LA_p}^{\mathrm{OBC}})}{A_p} \right\}.$$

Using the subadditivity of the measure, we obtain

$$\boldsymbol{P}\left(\bigcap_{p=1}^{\infty} \bar{\mathcal{N}}_p^{\mathrm{OBC}} \right) = \boldsymbol{P}\left\{ \liminf_{A\to\infty} \frac{W_{LA}^{\mathrm{OBC}}(\omega)}{A} \geqslant \frac{\boldsymbol{E}(W_{LA_p}^{\mathrm{OBC}})}{A_p},\ \forall p \in \mathbb{N} \right\} = 1$$

and, by proposition 4.1,

$$\boldsymbol{P}\left\{ \liminf_{A\to\infty} \frac{W_{LA}^{\mathrm{OBC}}(\omega)}{A} \geqslant v_0^2 L^2 \bar{g}_L^{\mathrm{OBC}} \right\} = 1.$$

For closed BC we can use similar arguments using the subadditivity of W_{LA}^{CBC} instead of the superadditivity of W_{LA}^{OBC}, and we obtain

$$\boldsymbol{P}\left\{ \limsup_{A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}(\omega)}{A} \leqslant v_0^2 L^2 \bar{g}_L^{\mathrm{CBC}} \right\} = 1.$$

Then, by lemma 4.1 and proposition 4.3, which implies $\bar{g}_L^{\mathrm{CBC}} = \bar{g}_L^{\mathrm{OBC}} \equiv \bar{g}_L$, we find that the limit

$$\lim_{A\to\infty} \frac{W_{LA}^{\mathrm{CBC,OBC}}(\omega)}{A} = v_0^2 L^2 \bar{g}_L \tag{31}$$

exists finite for every $\omega \in \Omega \backslash (\mathcal{N}_{\mathrm{sup}}^{\mathrm{CBC,OBC}} \bigcap \mathcal{N}_{\mathrm{inf}}^{\mathrm{CBC,OBC}})$, where we have introduced the following sets of zero measure

$$\mathcal{N}_{\mathrm{sup}}^{\mathrm{CBC}} = \left\{ \limsup_{A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}(\omega)}{A} > v_0^2 L^2 \bar{g}_L \right\}$$
$$\mathcal{N}_{\mathrm{inf}}^{\mathrm{CBC}} = \left\{ \liminf_{A\to\infty} \frac{W_{LA}^{\mathrm{CBC}}(\omega)}{A} < v_0^2 L^2 \bar{g}_L \right\}$$
$$\mathcal{N}_{\mathrm{sup}}^{\mathrm{OBC}} = \left\{ \limsup_{A\to\infty} \frac{W_{LA}^{\mathrm{OBC}}(\omega)}{A} > v_0^2 L^2 \bar{g}_L \right\}$$
$$\mathcal{N}_{\mathrm{inf}}^{\mathrm{OBC}} = \left\{ \liminf_{A\to\infty} \frac{W_{LA}^{\mathrm{OBC}}(\omega)}{A} < v_0^2 L^2 \bar{g}_L \right\}.$$

Since $\boldsymbol{P}(\mathcal{N}_{\mathrm{sup}}^{\mathrm{CBC,OBC}} \bigcap \mathcal{N}_{\mathrm{inf}}^{\mathrm{CBC,OBC}}) = 0$, the limit (31) exist almost surely and is finite both for open and closed BC. The thesis follows from theorem 4.1 and proposition 4.2.

Appendix. Theorems on subadditive and superadditive sequences

Theorem A.1 (Kingman [9]). Let $\{\xi_n\}$, $n \in \mathbb{N}$, be a subadditive sequence such that $\boldsymbol{E}(\xi_n) \geqslant -An$ for some positive constant A. Then, the limit

$$\xi = \lim_{n\to\infty} \frac{\xi_n}{n} \tag{A1}$$

exist finite almost surely and in the mean and $\boldsymbol{E}(\xi) = \gamma$, where

$$\gamma = \inf_n \frac{\boldsymbol{E}(\xi_n)}{n}. \tag{A2}$$

Moreover, the limit ξ can be represented as

$$\xi = \lim_{n\to\infty} \frac{\boldsymbol{E}(\xi_n|\mathrm{A})}{n} \tag{A3}$$

where A is the σ-algebra of the events invariant under the measure-preserving transformation on the probability space which defines the subadditive process. In particular, if A contains only events of probability 0 or 1, then $\xi = \gamma$.

The limit variable ξ is degenerate only in particular cases. For example, if $\xi_n = F_n(\eta_1, \eta_2, \ldots)$, where $\eta_1, \eta_2, \ldots$ are independent and identically distributed variables, then by the zero-one law the σ-algebra A is trivial and $\xi = \gamma$ [9]. Indeed, this is the case for the problem at hand, where the sequence ξ_n is given by the dissipation as a function of the volume of the region Λ and the variables $\eta_1, \eta_2, \ldots$ are the dissipations associated to the subvolumes of Λ.

Actually, in the case that the transformation is a translation one can show directly that ξ is degenerate, i.e. we have the following†.

Theorem A.2 (Bellisard et al [3]). Let $\{\xi_n\}$, $n \in \mathbb{N}$, be a superadditive sequence with measure-preserving translation τ on the probability space, such that $\boldsymbol{E}(\xi_n) \leqslant An$ for some positive constant A. Then, the limit

$$\xi = \lim_{n\to\infty} \frac{\xi_n}{n} \tag{A4}$$

exist finite almost surely and in the mean and $\xi = \gamma$, where

$$\gamma = \sup_n \frac{\boldsymbol{E}(\xi_n)}{n}. \tag{A5}$$

We also note the following lemma [9]:

Lemma Appendix .4. Let $\{a_n\}$, $n \in \mathbb{N}$ be a numeric sequence. Then

(i) $\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}$ if a_n is subadditive,

(ii) $\lim_{n\to\infty} \frac{a_n}{n} = \sup_n \frac{a_n}{n}$ if a_n is superadditive.

† The results obtained for superadditive sequences can be easily reformulated for subadditive sequences, and vice versa.

References

[1] See, for example, Shante V K S and Kirkpatrick S 1971 *Adv. Phys.* **20** 325
Kirkpatrick S 1973 *Rev. Mod. Phys.* **45** 574
[2] Bellisard J and Essoh C D 1989 *J. Phys. A: Math. Gen.* **22** 4537
[3] Bellisard J *et al* 1995 *Stochastic Processes, Physics and Geometry (Locarno)* vol II, ed S Albeverio *et al* (Singapore: World Scientific) p 107.
[4] Bruno O and Golden K 1990 *J. Stat. Phys* **61** 365
[5] Golden K 1990 *Phys. Rev. Lett.* **65** 2923
[6] Golden K 1992 *Commun. Math. Phys.* **143** 467
[7] Guerra F *et al* 1976 *Ann. Inst. H. Poincaré* **25** 231
[8] Ruelle D 1969 *Statistical Mechanics* (New York: Benjamin)
[9] Kingman J F C 1968 *J. R. Stat. Soc.* B **30** 499
[10] Ruelle D 1979 *Publ. IHES* **50** 275
[11] Shiryayev A N 1984 *Probability* (Berlin: Springer)